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SLOW MOTION OF A GRANULAR LAYER ON AN INCLINED PLANE

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The shape of the free surface of a layer of granular material moving on an inclined plane is studied on the basis of a model of a non-Newtonian fluid with a nonlinear relation between the stress tensor and the shear rate of the flow. For small but finite elevations of the free surface, the governing equations are reduced to a quasilinear Burgers equation. Results of a numerical solution are presented for the case of arbitrary elevations.

Granular media consisting of a large number of solid particles are abundant in nature and practical human activity: avalanches, sand storms, mudflows, powder metallurgy, chemical technology, storage and transportation of grain. Therefore, studies of the character of granular flows are important for both basic science and practical applications. It is commonly believed that these materials can be modeled adequately using the concepts and methods of continuum mechanics. Two different limiting flow regimes are usually distinguished for these media: a *quasistatic* regime, which corresponds to large densities and small shear rates, and an *inertial* regime, which corresponds to smaller densities and larger shear rates. In the first regime, all granules are always in close contact with their nearest neighbors and their motion is determined by Coulomb friction. In the second regime, there are always gaps between the granules and the interaction is conditioned by inelastic collisions. A description of the inertial regime is generally based on the laws of conservation of mass, momentum, and sometimes energy of random motion of granules (see, for example, [1]).

Careful and well-documented experiments [2] showed that the shear stress in inertial granular shear flows is proportional to the shear rate squared, in contrast to the ordinary viscous (Newtonian) fluid, for which this relationship is linear. Therefore, it is reasonable to regard granular materials as a non-Newtonian fluid (medium) with a nonlinear relationship between the stress tensor and shear rate of the flow.

We consider a layer of granular material with a free surface that moves on a rough plane. We study this two-dimensional flow within the framework of the non-Newtonian fluid model. Having directed the x axis along the inclined plane and the y axis across the plane, we write the governing equations as

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x} + \rho g\sin\alpha + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y},$$
$$\rho\frac{dv}{dt} = -\frac{\partial p}{\partial y} - \rho g\cos\alpha + \frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

We make the following assumptions:

(1) the longitudinal scale L_0 is much larger than the transverse scale H_0, i.e., $\varepsilon = H_0/L_0 \ll 1$;

(2) the transverse velocity v is much smaller than the longitudinal velocity u, i.e., $v \ll u$, but $\partial v/\partial y \sim \partial u/\partial x$;

(3) the flow is slow, which is valid for some natural granular flows that slip down an inclined plane, for instance, avalanches and glaciers, and the acceleration dv/dt can be ignored [3];

(4) $\partial\sigma_{xx}/\partial x \ll \partial\tau_{xy}/\partial y$ in the first equation;

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(5) $\partial\tau_{yx}/\partial x$ and $\partial\sigma_{yy}/\partial y \ll \partial p/\partial y$ and $\rho g\cos\alpha$ in the second equation;

(6) the shear stress $\tau_{xy} = \tau_{yx}$ is proportional to $|\partial u/\partial y|^{n-1}\partial u/\partial y$.

The exponent $n = 2$ corresponds to a granular medium in the inertial regime, as in experiments [2], $n = 1/2$ refers to a pseudoplastic medium, and $n = 1$ corresponds to a non-Newtonian viscous liquid.

Under these assumptions, system (1) takes the form

$$\rho^{-1}\frac{\partial p}{\partial x} = g\sin\alpha + \nu_n\partial\Big(\Big|\frac{\partial u}{\partial y}\Big|^{n-1}\frac{\partial u}{\partial y}\Big)\Big/\partial y, \quad \rho^{-1}\frac{\partial p}{\partial y} = -g\cos\alpha, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

Here ν_n is the kinematic viscosity that corresponds to the exponent of the model n.

System (2) should be supplemented by boundary conditions in the form

$$u = v = 0 \quad \text{at the bottom of the layer} \quad (y = 0); \tag{3}$$

$$p = 0, \quad \tau_{xy} = 0, \quad H_t + uH_x = v \quad \text{on the free surface} \quad [y = H(x,t)]. \tag{4}$$

Since the shear stress τ_{xy} is proportional to $(\partial u/\partial y)^n$, the equality of this shear stress to zero on the free surface is equivalent to the condition $\partial u/\partial y = 0$ for $y = H(x,t)$.

According to the second equation of system (2), the pressure is hydrostatic: $p = \rho g(H - y)\cos\alpha$. Hence, $p_x/\rho = gH_x\cos\alpha$. Substituting the latter relation into the first equation of system (2) and integrating it with the boundary conditions $\partial u/\partial y = 0$ for $y = H(x,t)$ and $u = 0$ for $y = 0$, we obtain the longitudinal-velocity profile

$$u = n(n+1)^{-1}A^{1/n}B^{1/n}(H^{1+1/n} - (H - y)^{1+1/n}),$$

where $A = g\cos\alpha/\nu_n$ and $B = \tan\alpha - H_x$. For flow regions with $H_x > 0$, this analysis is valid for even n when $H_x < \tan\alpha$. Near the bottom of the layer ($y = 0$) the longitudinal-velocity profile is a linear function of the transverse coordinate: $u = A^{1/n}B^{1/n}H^{1/n}y$. Using a continuity equation with the boundary condition $v = 0$ for $y = 0$, we obtain the transverse-velocity profile.

$$v = (n+1)^{-1}A^{1/n}B^{1/n}\{(B^{-1}H^{1+1/n}H_{xx} - (n+1)H^{1/n}H_x)y + n(H^{1+1/n} - (H-y)^{1+1/n})H_x - n(2n+1)^{-1}B^{-1}(H^{2+1/n} - (H-y)^{2+1/n})H_{xx}\}.$$

If we now substitute the values of the longitudinal- and transverse-velocity components on the free surface $[y = H(x,t)]$ into the kinematic boundary condition, we obtain the sought equation for the shape of the free surface of the granular layer under consideration:

$$H_t + A^{1/n}B^{1/n}H^{1+1/n}H_x = (2n+1)^{-1}A^{1/n}B^{1/n-1}H^{2+1/n}H_{xx}. \tag{5}$$

We now convert to dimensionless variables, for which we introduce the length scale L_0, thickness scale H_0, and time scale $t_0 = (\nu_n/gH_0)^{1/n}$ (H_0 is the undisturbed thickness of the layer). Then Eq. (5) can be written as

$$H_t + \varepsilon(\cos\alpha)^{1/n}(\tan\alpha - \varepsilon H_x)^{1/n}H^{1+1/n}H_x = \varepsilon^2(2n+1)^{-1}(\sin\alpha)^{1/n}\cot\alpha H^{2+1/n}H_{xx}. \tag{6}$$

The shape of the free surface changes as a result of nonlinear convective transfer and nonlinear diffusion.

For the case of small but finite elevations of the free surface above the undisturbed level $H = 1 + h$ ($h \ll 1$), Eq. (6) becomes

$$h_t + \varepsilon(\sin\alpha)^{1/n}(1 + (n+1)n^{-1}h)h_x = \varepsilon^2(2n+1)^{-1}(\sin\alpha)^{1/n}\cot\alpha h_{xx}.$$

This equation can be written in a simpler form if we use a coordinate system moving at a constant velocity $c = \varepsilon(\sin\alpha)^{1/n}$, namely

$$h_t + ahh_x = bh_{xx}, \quad a = \varepsilon(\sin\alpha)^{1/n}n^{-1}(n+1), \quad b = \varepsilon^2(2n+1)^{-1}(\sin\alpha)^{1/n}\cot\alpha. \tag{7}$$

Equation (7) is a well-known quasilinear Burgers equation that contains quadratic nonlinearity and viscosity with a constant coefficient. As $b \to 0$, the solution of Eq. (7) demonstrates an increase in the slope of the front of

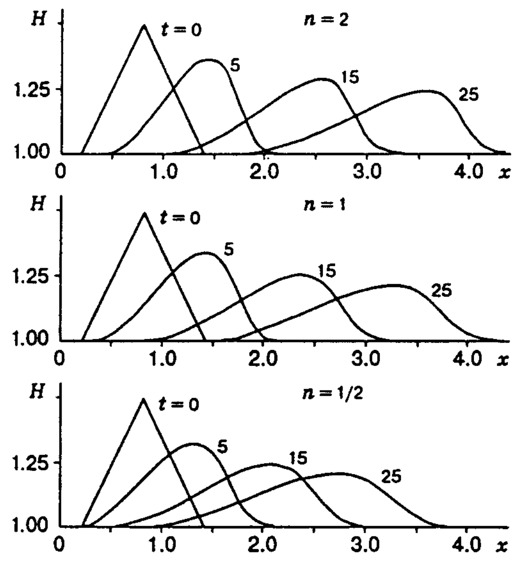

Fig. 1

disturbances with a discontinuity formed over a finite period of time. When the viscosity is different from zero, the increase in the slope of the front is compensated by the smearing effect of diffusion, which is proportional to h_{xx}, leading to formation of a shock wave with a constant width $\Delta \approx b/U$, where $U = (a/2)(h_{-\infty} + h_{\infty})$ is the wave velocity.

To analyze the evolution of the shape of the free surface, we solved Eq. (6) numerically using an explicit difference scheme with one-side differences for convective transfer (for the sake of stability) and central differences for diffusion. The scheme

$$H_t + F(H)H_x = G(H)H_{xx},$$

$$H_i^{m+1} = H_i^m - \frac{\delta t}{\delta x} F_i^m (H_i^m - H_{i-1}^m) + \frac{\delta t}{\delta x^2} G_i^m (H_{i+1}^m - 2H_i^m + H_{i-1}^m)$$

is monotonous, has an approximation error $O(\delta t, \delta x)$, and, hence, exhibits numerical diffusion. At the same time, since the problem is one-dimensional, we could take very small steps in time and coordinate, so that the numerical diffusion is negligibly small.

The computational domain consisted of four units ($x = 0$–4). For simplicity, the initial disturbance of the free surface was specified in the form of an isosceles triangle of height H_1 and width such that the aforementioned condition $H_x < \tan\alpha$ was valid.

Figure 1 shows profiles of the free surface of a granular layer ($n = 2$) at times $t = 5t_0$, $15t_0$, and $25t_0$ after the initial triangle of height $H_1 = 0.5H_0$ suddenly became free. The angle between the inclined plane and the horizon line is 45°, and the ratio of the transverse and longitudinal scales is $\varepsilon = 0.1$. It can be seen that a compression wave with an almost constant width of the front forms with time. This wave moves at an almost constant velocity, calculated from the displacement of the point with a maximum slope of the profile $|H_x|$.

The positions and values of the maximum height of the free surface at various times are shown in Table 1 for $n = 2$ and a wave velocity $U = 0.11$. The wave-front width, which is due to the competition of the nonlinearity and diffusion processes, is directly proportional to the diffusivity and inversely proportional to the wave amplitude. It is equal to $\approx$0.44 in dimensionless variables and to $\approx 4.4H_0$ in dimensional variables. To determine how the flow character depends on the model of the medium, i.e., on n, we conducted calculations that corresponded to a viscous Newtonian fluid ($n = 1$) and a pseudoplastic medium ($n = 1/2$) with the same values of steps in time δt and coordinate δx, aspect ratio ε, and initial disturbance amplitude H_1.

TABLE 1

t/t_0	H_{max}	x_*	H_{max}	x_*	H_{max}	x_*
	$n = 2$, $U = 0.11$		$n = 1$, $U = 0.095$		$n = 1/2$, $U = 0.08$	
5	1.36	1.44	1.34	1.39	1.32	1.31
15	1.28	2.53	1.25	2.35	1.24	2.09
25	1.24	3.55	1.21	3.26	1.20	2.78

Profiles of the free surface are presented in Fig. 1, and the coordinates and maximum heights are listed in Table 1 for $n = 1$ and $U = 0.095$ and $n = 1/2$ and $U = 0.08$. The diffusion coefficient, i.e., the coefficient at H_{xx}, in Eq. (6) is proportional to $(\sin\alpha)^{1/n}(2n+1)^{-1}H^{2+1/n}$. Comparison of the values of these coefficients for various values of n shows that for $n = 2$ the diffusivity is minimum, and for $n = 1/2$ it is maximum. Thus, for $t/t_0 \leqslant 25$ and $n = 2$ the mutual effects of nonlinearity and diffusion compensate for each other and a profile of constant width is formed, while for $n = 1/2$ the width of the profile is unstable, increasing from $\Delta = 0.44$ for $t = 15t_0$ to $\Delta = 0.58$ for $t = 25t_0$.

In conclusion, we note that since the inertial terms were ignored, we studied a quasisteady case. It would be of interest to consider the stability of these solutions, for example, in the same manner as was done for a Newtonian fluid (see [4, 5] and the papers cited there).

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